Approximate Analytic Solution to the Lotka Volterra Predator Prey Differential Equations Model

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Abstract

The paper provides an approximate analytic solution to the Lotka Volterra predator-prey differential equations by symbolic regression as suggested by Padua et al. (2018). The approximate analytic solution can be made as close as desired to the actual analytic solution involving complicated Lambert W functions as derived by Evans and Findley (2017). As a side result, the symbolic regression approach also provides an approximation to the otherwise less tractable Lambert W integral equation.

Keywords: Lotka-Volterra, finite difference method, symbolic regression, Lambert W function

1.0 Introduction

One of the famous differential equations that possess several applications is the Lotka Volterra (LV) predator prey model given by:

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha x_1 - \beta x_1 x_2 \qquad (1) \\ \frac{dx_2}{dt} &= -\gamma x_2 + \delta x_1 x_2 \end{aligned}$$

where $x_1(t)$ is the number of preys at time; $x_2(t)$ is the number of predator at time (t). Equation (1) is autonomous in the sense that the right-hand side is not explicitly expressed in terms of time. Since the publication of Lotka Volterra predator prey model (Volterra, 1926), it has been used to model chemical reactions and phenomena in other fields of the sciences such as applications in the field of neural networks (Noonburg, 1989), epidemiology (Roussel, 1997), mode-specific coupling in lasers (Abate, 1968).

Despite the growing number of Lotka Volterra applications, there is no closed form that analytic solution to this predator-prey model that exists in the literature. The equation (1) is known to be conservative which implies that the solutions must be periodic but the analytic forms of the solutions remain unknown. There have been many attempts to derive the analytic solutions to the LV model. The easiest route to solve equation (1) is to apply numerical integration. The numerical integration of an equation generates data which can then be plotted allowing for the visualization of solution curves.

More recently, Padua et al. (2018) attempted to apply symbolic regression to determine an approximate analytic solution to a first order initial value problem. In Padua et al.

(2018), finite difference method was applied to ordinary differential equations with initial values (IV). The recursive relation for $\{u_n\}$ generated the pairs $\{(t_0, u_n), (t_1, u_1), \ldots, (t_n, u_n)\}$ which satisfy the IVP:

$$\begin{aligned} u^{'}(t) &= f(t, u((t)), & u(0) = u_0 \\ u_n &= u_{(n-1)} + f(h, u_{(n-1)})h \end{aligned} \tag{2}$$

where h is the step size. The pairs $\{(t_i, u_i)\}_{i=0}^{n}$ are then used as inputs to a symbolic regression model which returns $\{u(t)\}$ in analytic form. This paper obtains approximate analytic solutions to the Lotka-Volterra equations using the method suggested by Padua et al. (2018).

2.0 The Lotka-Volterra Model

Lotka-Volterra model in (1) is one of the simplest models of predator-prey interactions. The model was developed independently by Lotka (1925) and Volterra (1926).

The model contains two variables (x_1, x_2) and several parameters: intrinsic

$$\begin{aligned} x_1 &= density\,of\,prey \\ x_2 &= density\,of\,predators \\ \alpha &= intrinsic\,rate\,of\,prey\,population\,increase \\ \beta &= predation\,rate\,coefficient \\ \delta &= reproduction\,rate\,of\,predators\,per\,1\,prey\,eaten \\ \gamma &= predator\,mortality\,rate \end{aligned}$$

Solutions to the Differential Equations

Now, let us derive the solution of the above differential equation. If we let $x = x_1$ and $y = x_2$, the LV model can be rewritten as:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x + \beta xy \\ \frac{dy}{dt} &= -\gamma y + \delta xy \end{aligned} \tag{3}$$

The equations have periodic solutions and do not have a simple expression in terms of the usual trigonometric functions. We rewrite the equations as:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x + \beta xy = x(\alpha + \beta y) \\ \frac{dy}{dt} &= -\gamma y + \delta xy = y(-\gamma + \delta x) \end{aligned} \tag{4}$$

From the first equation:

$$\frac{\dot{x}}{x} = \alpha - \beta y \Rightarrow \beta y = \alpha - \frac{\dot{x}}{x} \tag{5}$$

We subscribe it into the second:

$$
\begin{aligned}
\beta y &= \beta y(-\gamma + \delta x) && (6)\\
\frac{d}{dt}\frac{\dot{x}}{x} + \Big(\alpha - \frac{\dot{x}}{x}\Big)(-\gamma + \delta x) &= 0\\
\frac{x\ddot{x} - \dot{x}^2}{x^2} + \Big(\alpha - \frac{\dot{x}}{x}\Big)(-\gamma + \delta x) &= 0\\
x\ddot{x} - \dot{x}^2 + (\alpha x^2 - x\dot{x})(\delta x - \gamma) &= 0
\end{aligned}
$$

Let $\dot{x} = x_1$, then $\ddot{x} = x_1 \frac{dx_1}{dt}$,

$$
x_1 x x_1' - x_1^2 + (\alpha x^2 - x x_1)(\delta x - \gamma) = 0 \tag{7}
$$

Divide by x^2:

$$
\frac{x_1}{x} x_1' - \frac{x_1^2}{x^2} + (\alpha - \frac{x_1}{x})(\delta x - \gamma) = 0 \tag{8}
$$

Let $x_1 = x x_2$:

$$
\begin{aligned}
x_2(x_2 + x x_2') - x_2^2 + (\alpha - x_2)(\delta x - \gamma) &= 0\\
x x_2 \frac{dx_2}{dx} + (\alpha - x_2)(\delta x - \gamma) &= 0\\
\frac{x_2 dx_2}{\alpha - x_2} + \Big(\delta - \frac{\gamma}{x}\Big) dx &= 0\\
-x_2 - \alpha \ln|x_2 - \alpha| + \delta x - \gamma \ln x &= C && (9)
\end{aligned}
$$

But $x_2 = \frac{x_1}{x} = \frac{\dot{x}}{x} = \alpha - \beta\gamma$ ($x_2 \leq \alpha$ always) and $x_{2_0} = \alpha - \beta\gamma_0$

$$
\begin{aligned}
-x_{2_0} - \alpha \ln|x_{2_0} - \alpha| + \delta x_0 - \gamma \ln x_0 &= C && (10)\\
-x_{2_0} - \alpha \ln(\beta y_0) + \delta x_0 - \gamma \ln x_0 &= C
\end{aligned}
$$

So,

$$
-x_{2_0} - \alpha \ln(\alpha - x_2) + \delta x - \gamma \ln x = C \tag{11}
$$

We may solve this equation using Lambert W function:

$$
x_2 = \alpha\Bigg[1 + W\Bigg(-\frac{1}{\alpha}\exp\Big(-\frac{1}{\alpha}\Big(\alpha - \delta x + \gamma \ln x + C\Big)\Big)\Bigg)\Bigg]. \tag{12}
$$

But $q = \frac{\dot{x}}{x}$ and

$$
\int \frac{dx/x}{1 + W\Bigg(-\frac{1}{\alpha}\exp\Big(-\frac{1}{\alpha}(\alpha - \delta x + \gamma \ln x + C)\Big)\Bigg)} = \int \alpha \, dt \tag{13}
$$

The Lambert W function, also called the product logarithm, is a set of functions namely the branches of the inverse relation of the function

$$f(z) = ze^z \tag{14}$$

and z is any complex number. In other words,

$$z = f^{-1}(ze^z) = W(ze^z) \tag{15}$$

By substituting the above equation in

$$z' = ze^z \tag{16}$$

we obtain the defining equation:

$$z' = W(z')e^{W(z')} \tag{17}$$

for any complex number z'.

Equation (12) has no closed form solution. However, Evans and Findley (2017) recently suggested a closed form solution to (12) for some specific values of the parameters. A linearization of the equations yields a solution similar to simple harmonic motion with the population of predators trailing that of prey by 90° in the cycle.

Parametric phase plot solutions

Parametric phase solutions consist of eliminating time from the two differential equations to produce a single differential equation consisting of prey and predator variables. These are then considered as orbits in the phase space without the time component. Thus,

$$\frac{dy}{dx} = -\frac{y}{x}\frac{\delta x - \gamma}{\beta y - \alpha}$$

relating the variables x and y. We note that the solutions to this equation are closed curves and is solvable by means of separation of variables technique:

$$\frac{\beta y - \alpha}{y}dy + \frac{\delta x - \gamma}{x}dx = 0$$

Integrating yields the implicit relationship

$$V = \delta x - \gamma \ln x + \beta y - \alpha \ln y$$

where V is an invariant and conserved for each curve. Evans and Findley (2017) exploited this particular property to solve equation (12).

Numerical Finite Difference Method

Let x_1 be the prey density and x_2 is the predator density, thus:

$$\begin{aligned} \frac{dx_1}{dt} &= ax_1\Big(1-\frac{x_1}{K}\Big)-\frac{\beta x_1 x_2}{1+\beta x_1} \\ \frac{dx_2}{dt} &= \gamma x_2\Big(1-\frac{x_2}{kx_1}) \end{aligned} \tag{18}$$

where $\beta x_1 x_2$ is the interaction rate between the species, $\frac{\beta x_1 x_2}{1+\beta x_1}$ is the effective rate of eating prey, γ is the mortality rate of the predators, where k are the carrying capacitance of each population.

Example: Wolf ate 100 rabbit out of 100 in 2 days. Then, Let $a=1$, $\gamma=0.5$, $K=500$, $k=0.82$, $\beta=1$, $x_1(0)=100$, $x_2(0)=100$.

The Eulers recursive relation is,

$$\begin{aligned} x_{1_i} &= x_{1_{i-1}} + x_{1_{i-1}}\, slope_{i-1}\Delta t \\ x_{2_i} &= x_{2_{i-1}} + x_{2_{i-1}}\, slope_{i-1}\Delta t \end{aligned}$$

This leads to the recurrence:

$$\begin{aligned} x_{1_i} &= x_{1_{i-1}} + \left(1x_{1_{i-1}}\Big(1-\frac{x_{1_{i-1}}}{500}\Big)-\frac{1x_{1_{i-1}}x_{2_{i-1}}}{1+1x_{1_{i-1}}}\right)(0.1) \\ x_{2_i} &= x_{2_{i-1}} + \left(0.5x_{2_{i-1}}\Big(1-\frac{x_{2_{i-1}}}{0.82x_{1_{i-1}}}\Big)\right)(0.1) \end{aligned} \tag{19}$$

3.0 Application of the Symbolic Regression Approach

Table 1. Results of First Ten Iterations

$h = 0.1$				$h = 0.01$				$h = 0.001$		
t	x1	x2		t0.01	x1	x2		t	x1	x2
0	100	100		0	100	100		0	100	100
0.1	98.09901	98.90244		0.01	99.8099	99.89024		0.001	99.98099	99.98902
0.2	96.19379	97.76753		0.02	99.61977	99.78012		0.002	99.96198	99.97805
0.3	94.28635	96.59695		0.03	99.4296	99.66962		0.003	99.94297	99.96706
0.4	92.37868	95.39239		0.04	99.2394	99.55876		0.004	99.92396	99.95608
0.5	90.47271	94.15565		0.05	99.04917	99.44754		0.005	99.90495	99.94509
0.6	88.57028	92.88852		0.06	98.85891	99.33595		0.006	99.88594	99.93409
0.7	86.67323	91.59287		0.07	98.66862	99.224		0.007	99.86692	99.92309
0.8	84.78328	90.27057		0.08	98.47831	99.11169		0.008	99.84791	99.91209
0.9	82.90214	88.92355		0.09	98.28798	98.99902		0.009	99.8289	99.90109
1	81.03143	87.55374		0.1	98.09763	98.886		0.01	99.80989	99.89008

The values were then entered into symbolic regression software with generated ordered pairs $\{(t_i, x_{1_i})\}_{i=0}^n$ for prey population and $\{(t_i, x_{2_i})\}_{i=0}^n$ for predator population. Let $t_i = ih$, $i = 0, 1, 2, \ldots n$, and with step size h. Results of the first little iteration are reproduced in Table1.

The summary statistics for the symbolic regression analysis with $h = 0.1$ is found in Table 2. We note the oscillatory characters of the solutions as expected. The prey population oscillates more often than the predator population. Furthermore, approximation error is larger for the prey population than for the predator.

Table 2: Summary Statistics for the Symbolic Regression Analysis with $h = 0.1$

Type	Mean Absolute Error	R^2	Equation
Prey ($x_1(t)$)	13.7946	0.9800	$x_1(t) = 102 + 146\cos(62.54t) + 50.74\cos(62.54t)^2 - 24.63\cos(0.18t) - 93.52\cos(62.59t) - 25.22\cos(62.54t)\cos(0.181t) - 87.85\cos(62.587t)\cos(62.54t)$
Predator ($x_2(t)$)	5.2949	0.9717	$x_2(t) = 58.8 + 52\sin(0.98 - 0.275t) + 37.5\sin(1.043 - 0.552t) + 51.3\sin(4.96 - 0.279t)^2\sin(0.98 - 0.275t)$

Figure 1 shows the graph of the solutions $x_1(t)$ and $x_2(t)$ when the step size is $h = 0.1$.

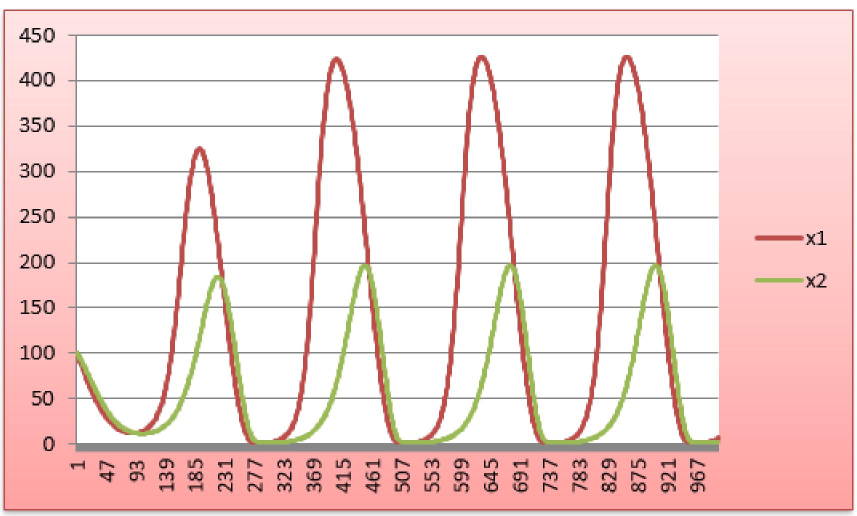

Figure 1: Plot of $x_1(t)$ and $x_2(t)$ with h = 0.1

Table 3 shows that summary statistics for the symbolic regression analysis with $h = 0.01$. Besides, approximation error is reduced for the prey population than for the predator.

Figure 2 shows the graph of the solutions $x_1(t)$ and $x_2(t)$ when the step size is reduced to $h = 0.01$.

Table 3: Summary Statistics for the Symbolic Regression Analysis with h =0.01

Type	Mean Absolute Error	R^2	Equation
Prey ($x_1(t)$)	11.0086	0.9868	$x_1(t) = 55.5 + 0.87t + 107\cos(2.68 - 0.29t)\cos(1.33 - 0.26t) - 31.3\cos(1.33 - 0.26t) - 104.7\cos(2.68 - 0.29t) - 1.35t\cos(1.33 - 0.26t)$
Predator ($x_2(t)$)	4.3718	0.9832	$x_2(t) = 28.5 + 72.5\sin(5.36 + 0.276t)\sin(4.82 + 0.275t) - 53.14\sin(5.29 + 0.276t) - 30.55\sin(4.82 + 0.275t)^2\sin(5.29 + 0.276t) - 0.276t\sin(4.82 + 0.275t)^2\sin(5.29 + 0.276t)$

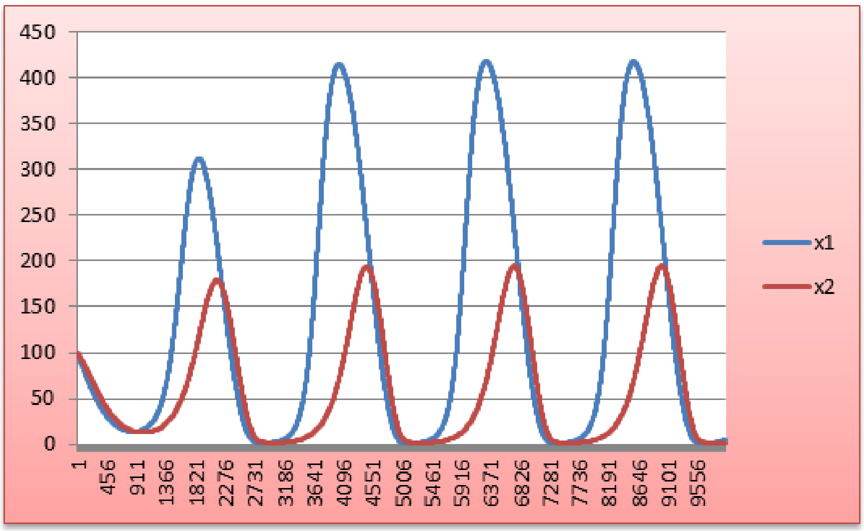

Figure 2: Plot of $x_1(t)$ and $x_2(t)$ with h = 0.01

Table 4 shows that summary statistics for the symbolic regression analysis with $h = 0.001$. Besides, approximation error is larger for the prey population than for the predator population.

Table 4: Summary Statistics for the Symbolic Regression Analysis with h =0.001

Type	Mean Absolute Error	R^2	Equation
Prey ($x_1(t)$)	10.3966	0.9857	$x_1(t) = 117 + 120\cos(0.295t) + 11.8\cos(0.594t) - 98.9\cos(0.697 - 0.259t) - 91.2\cos(0.295t)\cos(0.697 - 0.259t)$
Predator ($x_2(t)$)	4.3591	0.9831	$x_2(t) = 33.7 + 18.3\cos(0.27t) + 44.2\cos(0.776 + 0.276t) + 71.7\cos(0.275t)\cos(0.776 + 0.276t) + 26.6\cos(0.275t)^2\cos(0.776 + 0.276t) + 0.34t\cos(0.275t)^2\cos(0.776 + 0.276t)$

Figure 3 shows the graph of the solutions $x_1(t)$ and $x_2(t)$ when the step size is reduced to $h = 0.001$.

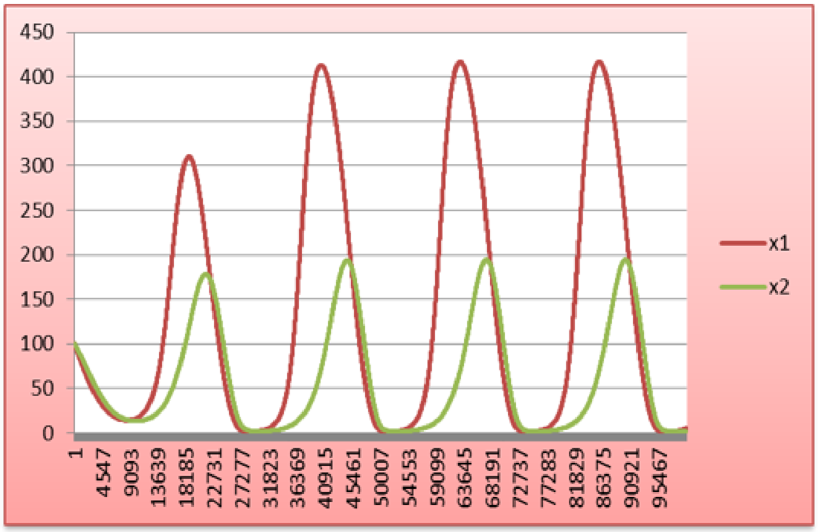

Figure 3: Plot of $x_1(t)$ and $x_2(t)$ over one full cycle with h = 0.001

Figure 4 shows the plot of $x_1(t)$ vs $x_2(t)$. Note the closed trajectory as expected.

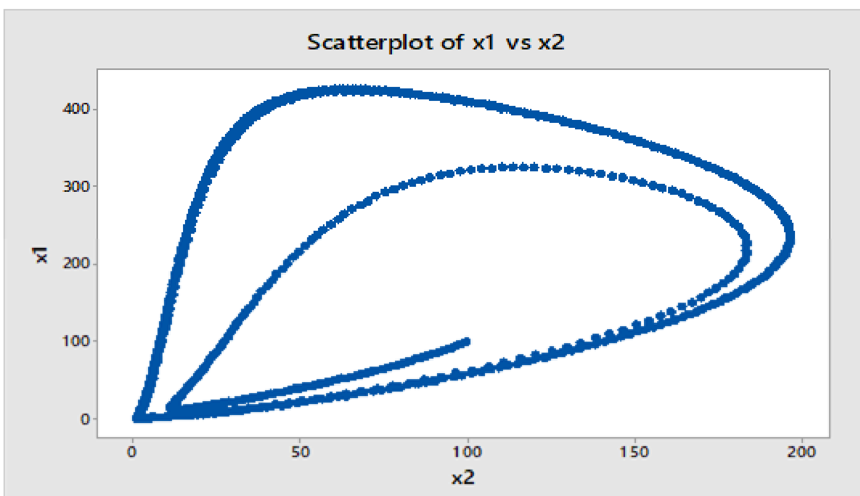

Figure 4: Plot of $x_1(t)$ against $x_2(t)$

We also analysed the difference between the number of prey $x_1(t)$ and the number of predator $x_2(t)$ at time t. If the difference becomes negative, then the prey population becomes extinct. However, if the difference stays positive, then the predator-prey populations continue to oscillate. Table 5 shows the descriptive statistics for this purpose with $h = 0.001$.

Since the differences lie between -43.11 and 370.12, we conclude that the predator prey populations continue to oscillate between small and large values.

Table 5: Descriptive Statistics for the difference $x_1(t)$ - $x_2(t)$.

Variable	N	N*	Mean	SE Mean	StDev
difference	1001	0	78.27	3.94	124.76

Minimum	Q1	Median	Q3	Maximum
-43.11	-3.66	6.49	152.97	370.12

Figure 6 shows the evolution of the differences over time.

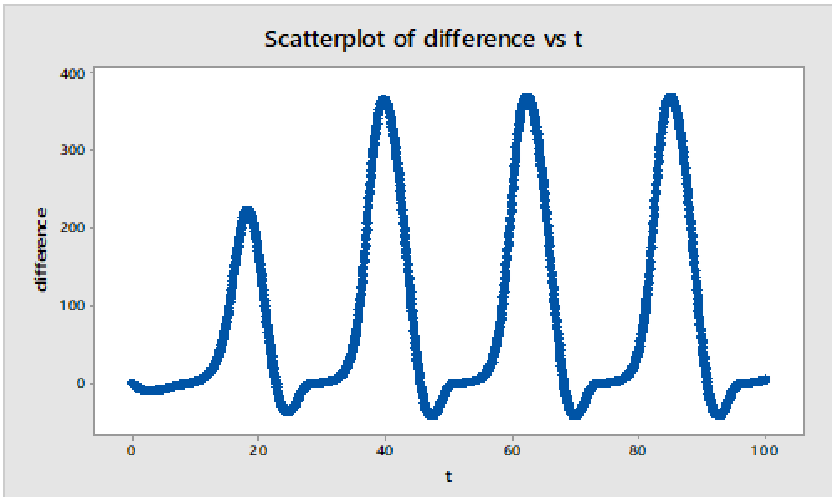

Figure 5: Evolution of the population differences over time

Main Result: The metric used was the smallest of Mean Absolute Error (MAE) and highest of R^2 Goodness of Fit. The solution for x_1 and x_2 are shown below:

$$
\begin{aligned}
x_1 &= 117 + 120\cos(0.295t) + 11.8\cos(0.594t) - 98.9\cos(0.697 - 0.259t) \\
&- 91.2\cos(0.295t)\cos(0.697 - 0.259t) \\
x_2 &= 33.7 + 18.3\cos(0.27t) + 44.2\cos(0.776 + 0.276t) + 71.7\cos(0.275t)\cos(0.776 + 0.276t) \\
&+ 26.6\cos(0.275t)^2\cos(0.776 + 0.276t) + 0.34t\cos(0.275t)^2\cos(0.776 + 0.276t)
\end{aligned}
$$

Conclusion

The symbolic regression approach provides a convenient means to determine an approximate solution to the Lotka-Volterra non-linear differential equations. Moreover, the approximate solution can be made as close as desired to the actual solution by making finer subdivisions of the interval of interest. Likewise, since the finite difference approach is a crude method for arriving at a numerical solution, faster convergence can be expected with more sophisticated Runge-Kutta numerical methods.

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