ON THE DISTRIBUTION OF THE PRODUCTS AND QUOTIENT OF INVERSE BURR DISTRIBUTED RANDOM VARIABLES BASED ON FGM COPULA

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ABSTRACT

In this article, Inverse Burr Distribution based on Farlie-Gumbel-Morgenstern copula is introduced. Derivations of exact distribution V = XY, W = X/Y, and Z = X/(X+Y) are obtained in closed form. Corresponding moment properties of these distributions are also derived. The expressions turn out to involve known special functions.

Keywords: *Inverse Burr distribution, Gauss Hypergeometric function, products, quotient of random variables.*

1.0 Introduction

The name copula was introduced by Sklar (1959) which means to connect or to join. Its sole purpose is to describe the interdependence of several random variables (Schmidt, 2007). A copula is a joint distribution function of the uniform marginals (Nelsen, 2003). When marginals are uniform, they are independent. This implies a flat probability density function and any deviation will indicate dependency (Balakrishnan and Lai, 2009).

The notion of copulas became increasingly popular. The main reason of this increased interest has to be found that it is useful in a variety of modeling situations and have been applied in a wide areas such as quantitative risk management (McNeil et al., 2005), econometric modeling (Patton, 2012), environmental modeling (Salvadori et al., 2007).

In this study, a Farlie-Gumbel-Morgenstern (FGM) copula is considered in constructing a bivariate pdf that accounts dependence between two random variables. Let $F_X(x)$ and $F_Y(y)$ be the distribution functions of the random variables X and Y, respectively, and θ, $-1 < \theta < 1$, then the joint probability density function or FGM copula density of X and Y is given by

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\left[1 + \theta\left(2F_X(x) - 1\right)\left(2F_Y(y) - 1\right)\right] \tag{1}$$

where $f_X(x)$ and $f_Y(y)$ are the pdf's of random variable X and Y, respectively. The parameter θ is known as the dependence parameter of X and Y.

The FGM copula was first proposed by Morgenstern (1956). According to Trivedi and Zimmer (2007), the FGM copula is a perturbation of the product copula; if the dependence parameter θ equals zero, then the FGM copula collapses to independence. It is attractive due to its simplicity. However, it is restrictive because this copula is only useful when dependence between the two marginals is modest in magnitude.

Several researchers studied on obtaining exact distributions on the sum, product and quotient of some known bivariate distributions (Nadarajah, 2005; Nadarajah & Espejo, 2006; Nadarajah & Kotz, 2007), and also Coelho and Mexia (2007) obtained the product and ratio of independent generalized gamma-ratio random variables. However, the researchers of this study used a bivariate Inverse Burr distribution based on FGM copula. As to our knowledge, there is still no research done with this marginal.

The paper is organized as follows. Section 2 is devoted on derivations of explicit expressions for the pdfs of, V = XY, W = X/Y, and Z = X/(X + Y), respectively, while section 3 is devoted in derivation of raw moments of all pdfs obtained in section 2.

The calculations of this paper involve several special functions. These include the incomplete beta function

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt,$$

and, the Gauss Hypergeometric function

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The following results which can be found in Nadarajah and Espejo (2006) are needed in the subsequent discussions.

LEMMA 1. *For any $\rho > \alpha > 0$,*

$$\int_0^\infty \frac{s^{\alpha-1}}{(s+z)^\rho}ds = z^{\alpha-\rho}B(\alpha,\rho-\alpha), \quad z \in \mathbb{R}, \tag{2}$$

where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

for $a > 0$ and $b > 0$ is the beta function.

LEMMA 2. *For $0 < \alpha < \rho + \lambda$,*

$$\int_0^\infty x^{\alpha-1}(x+y)^{-\rho}(x+z)^{-\lambda}dx = z^{-\lambda}y^{\alpha-\rho}B(\alpha,\rho+\lambda-\alpha)\,{}_2F_1\left(\alpha,\lambda;\rho+\lambda;1-\frac{y}{z}\right). \tag{3}$$

2.0 Probability Density Functions

If X and Y follows Inverse Burr distribution then the pdf has the form

$$f_X(x;\theta,\tau,\gamma) = \frac{\tau\gamma(\frac{x}{\theta})^{\tau\gamma}}{x[1+(\frac{x}{\theta})^{\gamma}]^{\tau+1}} \quad ; \quad x,\tau,\theta,\gamma > 0 \tag{4}$$

and

$$f_Y(y;\theta,\tau,\gamma) = \frac{\tau\gamma(\frac{y}{\theta})^{\tau\gamma}}{y[1+(\frac{y}{\theta})^{\gamma}]^{\tau+1}} \quad ; \quad y,\tau,\theta,\gamma > 0 \tag{5}$$

respectively, for positive values of x and y and the cdf of these distribution are known to be

$$F_X(x;\theta,\tau,\gamma) = \left(\frac{(\frac{x}{\theta})^{\gamma}}{1+(\frac{x}{\theta})^{\gamma}}\right)^{\tau} \quad ; \quad x,\tau,\theta,\gamma > 0 \tag{6}$$

and

$$F_Y(y;\theta,\tau,\gamma) = \left(\frac{(\frac{y}{\theta})^{\gamma}}{1+(\frac{y}{\theta})^{\gamma}}\right)^{\tau} \quad ; \quad y,\tau,\theta,\gamma > 0 \tag{7}$$

The following result is the joint pdf constructed from F-G-M copula using Inverse Burr distribution as marginals.

Theorem 2.1. *If X and Y follows Inverse Burr distribution, then the joint pdf of X and Y is given by*

$$\begin{aligned} f_{X,Y}(x,y;\tau,\theta,\gamma;\rho) = & \frac{\tau\gamma(\frac{x}{\theta})^{\tau\gamma}}{x[(1+(\frac{x}{\theta}))^{\gamma}]^{\tau}} \cdot \frac{\tau\gamma(\frac{y}{\theta})^{\tau\gamma}}{y[(1+(\frac{y}{\theta}))^{\gamma}]^{\tau}} \\ & \left[1+\rho\left(1-2\left(\frac{(\frac{x}{\theta})^{\gamma}}{1+(\frac{x}{\theta})^{\gamma}}\right)^{\tau}\right)\left(1-2\left(\frac{(\frac{y}{\theta})^{\gamma}}{1+(\frac{y}{\theta})^{\gamma}}\right)^{\tau}\right)\right] \end{aligned} \tag{8}$$

for $x, y > 0$ and $f_{X,Y}(x,y;\tau,\theta,\gamma;\rho) = 0$ elsewhere with parameters $\tau,\gamma,\theta > 0$ and $|\rho| \leq 1$.

Proof. Plugging-in Equations (4),(5),(6) and (7) in Equation (1), we have

$$\begin{aligned} f_{X,Y}(x,y;\tau,\theta,\gamma;\rho) = & \frac{\tau\gamma(\frac{x}{\theta})^{\tau\gamma}}{x[(1+(\frac{x}{\theta}))^{\gamma}]^{\tau}} \cdot \frac{\tau\gamma(\frac{y}{\theta})^{\tau\gamma}}{y[(1+(\frac{y}{\theta}))^{\gamma}]^{\tau}} \\ & \left[1+\rho\left(1-2\left(\frac{(\frac{x}{\theta})^{\gamma}}{1+(\frac{x}{\theta})^{\gamma}}\right)^{\tau}\right)\left(1-2\left(\frac{(\frac{y}{\theta})^{\gamma}}{1+(\frac{y}{\theta})^{\gamma}}\right)^{\tau}\right)\right] \end{aligned}$$

It can be shown that (8) is nonnegative.
To show that

$$\int_0^{\infty}\int_0^{\infty} f_{X,Y}(x,y;\tau,\theta,\gamma;\rho)dxdy = 1$$

Observed that

$$\int_0^{\infty} \frac{\tau\gamma(\frac{x}{\theta})^{\tau\gamma}}{x[(1+(\frac{x}{\theta}))^{\gamma}]^{\tau}}\left(1-2\left(\frac{(\frac{x}{\theta})^{\gamma}}{1+(\frac{x}{\theta})^{\gamma}}\right)^{\tau}\right)dx = 0$$

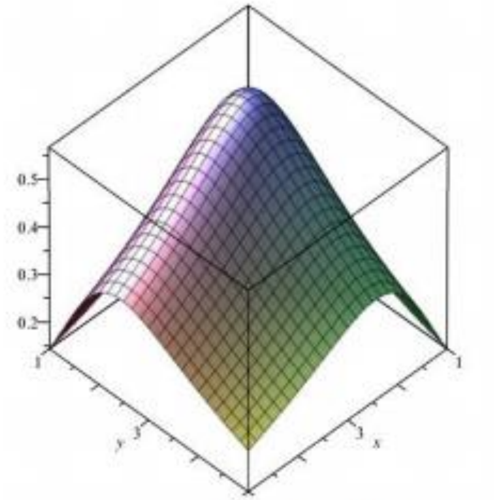

Figure 1: Graph of the pdf in (8)

Similarly,

$$\int_0^\infty \frac{\tau\gamma(\frac{y}{\theta})^{\tau\gamma}}{y[(1+(\frac{y}{\theta}))^{\gamma}]^{\tau}}\left(1-2\left(\frac{(\frac{y}{\theta})^{\gamma}}{1+(\frac{y}{\theta})^{\gamma}}\right)^{\tau}\right)dy = 0$$

then it follows that

$$\int_0^\infty \int_0^\infty f_{X,Y}(x,y;\tau,\theta,\gamma;\rho)dxdy = 1$$

Therefore, $f_{X,Y}(x,y;\theta,\tau,\gamma;\rho)$ is indeed a pdf . □

Figure 1 shows the graph of Equation (8) for specific values of $\gamma = 4$, $\theta = 2$, $\tau = 0.5$, and $\rho = 0.5$

Theorem 2.2. *If X and Y are jointly distributed according to Equation* (8)*, then the pdf of $V = XY$ is given by*

$$\begin{aligned} f_V(v;\theta,\tau,\gamma;\rho) = \gamma\tau^2\theta^{2\gamma}v^{-(1+\gamma)}\Bigg[&(1+\rho)B(\tau+1,\tau+1)_2F_1\left(\tau+1,\tau+1;2\tau+2;1-\frac{\theta^{2\gamma}}{v^{\gamma}}\right) \\ &-2\rho B(2\tau+1,\tau+1)_2F_1\left(2\tau+1,\tau+1;3\tau+2;1-\frac{\theta^{2\gamma}}{v^{\gamma}}\right) \\ &-2\rho B(\tau+1,2\tau+1)_2F_1\left(\tau+1,2\tau+1;3\tau+2;1-\frac{\theta^{2\gamma}}{v^{\gamma}}\right) \\ &+4\rho B(2\tau+1,2\tau+1)_2F_1\left(2\tau+1,2\tau+1;4\tau+2;1-\frac{\theta^{2\gamma}}{v^{\gamma}}\right)\Bigg] \end{aligned} \quad (9)$$

for $0 < v < \infty$.

Proof. From (8), the joint pdf of (X, V) = (X, XV) and applying the Rohatgi and Saleh well-known result (Theorem 3, p.139-140), becomes

$$f_V(v;\theta,\tau,\gamma;\rho) = \int_0^\infty x^{-1} f_{X,Y}(x, v/x;\theta,\tau,\gamma;\rho) \tag{10}$$

For simplicity, we write

$$f_V(v;\theta,\tau,\gamma;\rho) = (1+\rho)\Psi(1,1) - 2\rho\Psi(2,1) - 2\rho\Psi(1,2) + 4\rho\Psi(2,2) \tag{11}$$

where $\Psi(h,k)$, for $h,k \in \{1,2\}$ is defined as follows

$$\Psi(h,k) = \frac{(\tau\gamma)^2 v^{k\gamma\tau-1}}{\theta^{k\gamma\tau-\gamma}} \int_0^\infty x^{h\gamma\tau+\gamma-1}(x^\gamma+\theta^\gamma)^{-(h\tau+1)}(x^\gamma+(v/\theta)^\gamma)^{-(k\tau+1)}dx.$$

Substituting $u = x^\gamma$, the integral $\Psi(h,k)$ can be written as

$$\Psi(h,k) = \frac{\tau^2\gamma v^{k\gamma\tau-1}}{\theta^{k\gamma\tau-1}} \cdot \lim_{b\to\infty}\int_0^b u^{h\tau+1-1}(u+\theta^\gamma)^{-(h\tau+1)}(u+(v/\theta)^\gamma)^{-(k\tau+1)}du \tag{12}$$

By Lemma 2, Equation (12) reduces to

$$\begin{aligned}\Psi(h,k) &= \frac{\tau^2\gamma v^{k\gamma\tau-1}}{\theta^{k\gamma\tau-1}} \cdot \lim_{b\to\infty}\int_0^b u^{h\tau+1-1}(u+\theta^\gamma)^{-(h\tau+1)}(u+(v/\theta)^\gamma)^{-(k\tau+1)}du \\ &= \frac{\tau^2\gamma\theta^{2\gamma}}{v^{(1+\gamma)}}\beta(h\tau+1,k\tau+1)_2F_1(h\tau+1,k\tau+1;h\tau+k\tau+2;1-(\theta^{2\gamma}/v^\gamma))\end{aligned}$$

Thus, the integral in Equation (11) can be simplified as follows:

$$(1)\ \Psi(1,1) = \gamma\tau^2\theta^{2\gamma}v^{-(1+\gamma)}B(\tau+1,\tau+1)_2F_1\left(\tau+1,\tau+1;2\tau+2;1-\frac{\theta^{2\gamma}}{v^\gamma}\right)$$

$$(2)\ \Psi(2,1) = \gamma\tau^2\theta^{2\gamma}v^{-(1+\gamma)}B(2\tau+1,\tau+1)_2F_1\left(2\tau+1,\tau+1;3\tau+2;1-\frac{\theta^{2\gamma}}{v^\gamma}\right)$$

$$(3)\ \Psi(1,2) = \gamma\tau^2\theta^{2\gamma}v^{-(1+\gamma)}B(\tau+1,2\tau+1)_2F_1\left(\tau+1,2\tau+1;3\tau+2;1-\frac{\theta^{2\gamma}}{v^\gamma}\right)$$

$$(4)\ \Psi(2,2) = \gamma\tau^2\theta^{2\gamma}v^{-(1+\gamma)}B(2\tau+1,2\tau+1)_2F_1\left(2\tau+1,2\tau+1;4\tau+2;1-\frac{\theta^{2\gamma}}{v^\gamma}\right)$$

And by substituting the results of (1)-(4) to Equation (11), the results follow. □

Figure 2 shows the graph of Equation (9) for $\gamma = 1, 2$. Each plot contains three curves corresponding to selected values of θ and τ. The effect of the parameters is evident.

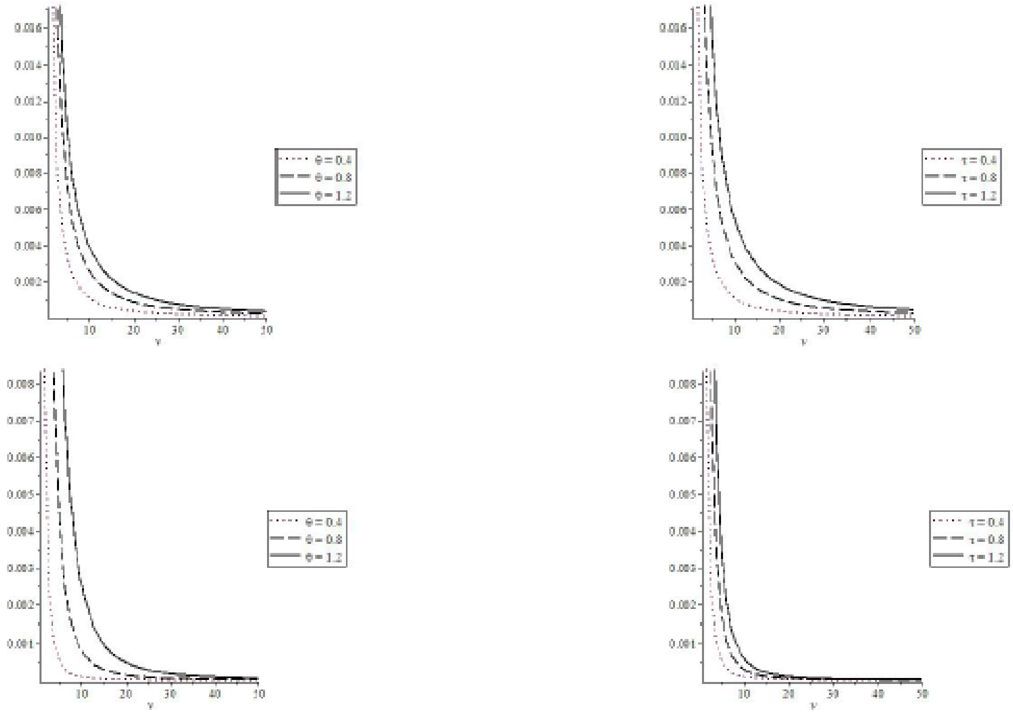

Figure 2: Graph of the pdf in (9)

Theorem 2.3. *If X and Y are jointly distributed according to 8, then the pdf of $W = X/Y$ is given by*

$$\begin{aligned} f_W(w;\theta,\tau,\gamma;\rho) = \tau^2\gamma\Bigg[&(1+\rho)w^{-(\gamma\tau+1)}B(2\tau,2)\,{}_2F_1\left(2\tau,\tau+1;2\tau+2;1-\left(\frac{\theta}{w^2}\right)^\gamma\right)\\ &-2\rho w^{-(\gamma\tau+1)}B(3\tau,2)\,{}_2F_1\left(3\tau,\tau+1;3\tau+2;1-\left(\frac{\theta}{w^2}\right)^\gamma\right)\\ &-2\rho w^{-(2\gamma\tau+1)}B(3\tau,2)\,{}_2F_1\left(3\tau,2\tau+1;3\tau+2;1-\left(\frac{\theta}{w^2}\right)^\gamma\right)\\ &+4\rho w^{-(2\gamma\tau+1)}B(4\tau,2)\,{}_2F_1\left(4\tau,2\tau+1;4\tau+2;1-\left(\frac{\theta}{w^2}\right)^\gamma\right)\Bigg] \end{aligned} \tag{13}$$

for $0 < w < \infty$.

Proof. From (8), the joint pdf of $(X, Y) = \left(X, \frac{X}{Y}\right)$ and applying the Rohatgi and Saleh well-known result (Theorem 3, p.139-140), becomes

$$f_W(w;\theta,\tau,\gamma;\rho) = \int_0^\infty y f_{X,Y}(yw, y;\theta,\tau,\gamma;\rho)dy \tag{14}$$

For simplicity, we write

$$f_W(w;\theta,\tau,\gamma;\rho) = (1+\rho)\Psi(1,1) - 2\rho\Psi(2,1) - 2\rho\Psi(1,2) + 4\rho\Psi(2,2) \tag{15}$$

where $\Psi(h,k)$, for $h,k \in \{1,2\}$ is defined as follows

$$\Psi(h,k) = (\tau\gamma)^2 w^{-(\gamma+1)}\theta^{2\gamma}\int_0^\infty y^{h\gamma\tau+k\gamma\tau-1}\left(y^\gamma+\left(\frac{\theta}{w}\right)^\gamma\right)^{-(h\tau+1)}\left(y^\gamma+\theta^\gamma\right)^{-(k\tau+1)}dy.$$

Substituting $u = y^\gamma$, the integral $\Psi(h,k)$ can be written as

$$\Psi(h,k) = \tau^2\gamma w^{-(\gamma+1)}\theta^{2\gamma}\cdot\lim_{b\to\infty}\int_0^b u^{h\tau+k\tau-1}\left(u+\left(\frac{\theta}{w}\right)^\gamma\right)^{-(h\tau+1)}\left(u+\theta^\gamma\right)^{-(k\tau+1)}du \tag{16}$$

By Lemma 2, Equation (16) reduces to

$$\Psi(h,k) = \tau^2\gamma w^{-(k\gamma\tau+1)}B(h\tau+k\tau,2)2F_1\left(h\tau+k\tau,k\tau+1;h\tau+k\tau+2;1-\left(\frac{\theta}{w^2}\right)^\gamma\right)$$

Thus, the integral in Equation (15), can be simplified as follows:

$$(1)\ \Psi(1,1) = \tau^2\gamma w^{-(\gamma\tau+1)}B(2\tau,2)\quad {}_2F_1\left(2\tau,\tau+1;2\tau+2;1-\left(\frac{\theta}{w^2}\right)^\gamma\right)$$

$$(2)\ \Psi(2,1) = \tau^2\gamma w^{-(\gamma\tau+1)}B(3\tau,2)\quad {}_2F_1\left(3\tau,\tau+1;3\tau+2;1-\left(\frac{\theta}{w^2}\right)^\gamma\right)$$

$$(3)\ \Psi(1,2) = \tau^2\gamma w^{-(2\gamma\tau+1)}B(3\tau,2)\quad {}_2F_1\left(3\tau,2\tau+1;3\tau+2;1-\left(\frac{\theta}{w^2}\right)^\gamma\right)$$

$$(4)\ \Psi(2,2) = \tau^2\gamma w^{-(2\gamma\tau+1)}B(4\tau,2)\quad {}_2F_1\left(4\tau,2\tau+1;4\tau+2;1-\left(\frac{\theta}{w^2}\right)^\gamma\right)$$

And by substitution of (1)-(4) to Equation (15), the results follow. □

Figure 3 shows the graph of Equation (13) for $\gamma = 1, 2$. Each plot contains three curves corresponding to selected values of θ and τ. The effect of the parameters is evident.

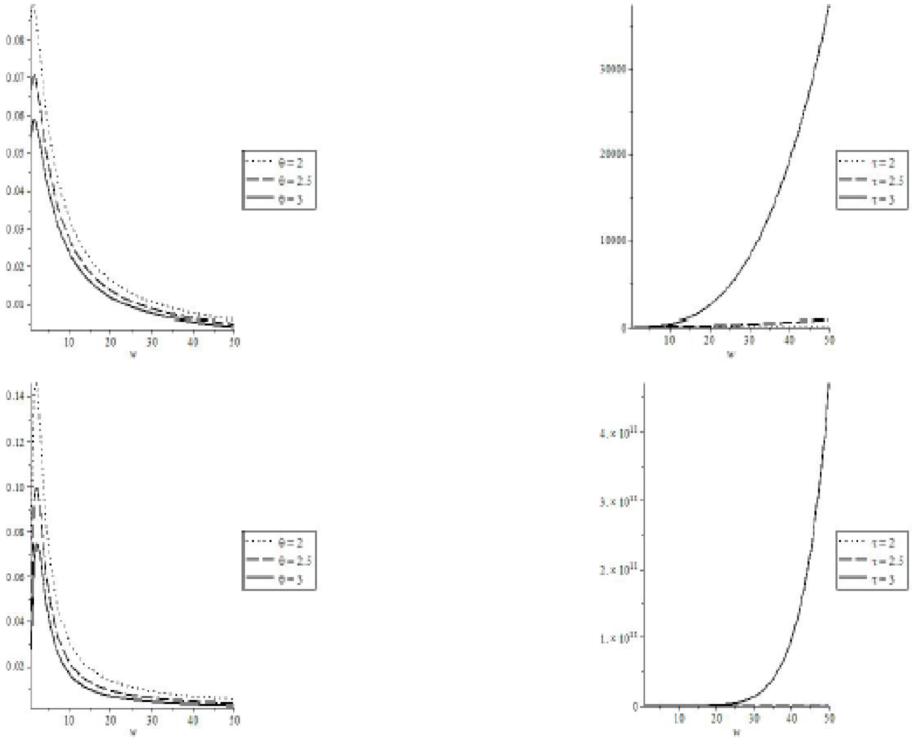

Figure 3: Graph of the pdf in (13)

Theorem 2.4. *If X and Y are jointly distributed according to Equation* (8), *then the pdf of $Z = X/(X+Y)$ is given by*

$$\begin{aligned} f_Z(z;\theta,\tau,\gamma,\rho) = \tau^2\gamma\Bigg[&(1+\rho)z^{-(\tau\gamma+1)}(1-z)^{\tau\gamma-1}B(2\tau,2)\,{}_2F_1\left(2\tau,\tau+1;2\tau+2;1-\left(\frac{1-z}{z}\right)^{\gamma}\right) \\ &-2\rho z^{-(\tau\gamma+1)}(1-z)^{\tau\gamma-1}B(3\tau,2)\,{}_2F_1\left(3\tau,\tau+1;3\tau+2;1-\left(\frac{1-z}{z}\right)^{\gamma}\right) \\ &-2\rho z^{-(2\tau\gamma+1)}(1-z)^{2\tau\gamma-1}B(3\tau,2)\,{}_2F_1\left(3\tau,2\tau+1;3\tau+2;1-\left(\frac{1-z}{z}\right)^{\gamma}\right) \\ &+4\rho z^{-(2\tau\gamma+1)}(1-z)^{2\tau\gamma-1}B(4\tau,2)\,{}_2F_1\left(4\tau,2\tau+1;4\tau+2;1-\left(\frac{1-z}{z}\right)^{\gamma}\right)\Bigg] \end{aligned} \tag{17}$$

for $0 < z < 1$.

Proof. Consider the transformation: (X, Y) → (R, Z) = $\left(X+Y, \frac{X}{X+Y}\right)$ so that

$$f_{R,Z}(r, z; \theta, \tau, \gamma; \rho) = f_Z(z; \theta, \tau, \gamma; \rho) \tag{18}$$

Note that the jacobian of transformation is r, for simplicity, we write

$$f_Z(z; \theta, \tau, \gamma; \rho) = (1+\rho)\Psi(1,1) - 2\rho\Psi(2,1) - 2\rho\Psi(1,2) + 4\rho\Psi(2,2) \tag{19}$$

where $\Psi(h,k)$, for $h, k \in \{1,2\}$ is defined as follows

$$\Psi(h,k) = (\tau\gamma)^2 z^{-(1+\gamma)}(1-z)^{-(1+\gamma)}\theta^{2\gamma}\int_0^\infty r^{h\tau\gamma+k\tau\gamma-1}\left(r^\gamma + \left(\frac{\theta}{z}\right)^\gamma\right)^{-(h\tau+1)} \left(r^\gamma + \left(\frac{\theta}{1-z}\right)^\gamma\right)^{-(k\tau+1)} dr.$$

Substituting $u = x^\gamma$, the integral $\Psi(h,k)$ can be written as

$$\Psi(h,k) = \tau^2\gamma z^{-(1+\gamma)}(1-z)^{-(1+\gamma)}\theta^{2\gamma} \cdot \lim_{b\to\infty}\int_0^b u^{h\tau+k\tau-1}\left(u + \left(\frac{\theta}{z}\right)^\gamma\right)^{-(h\tau+1)} \left(u + \left(\frac{\theta}{1-z}\right)^\gamma\right)^{-(k\tau+1)} du \tag{20}$$

By Lemma 2, Equation (20) reduces to

$$\Psi(h,k) = \tau^2\gamma z^{-(k\tau\gamma+1)}(1-z)^{k\tau\gamma-1}\beta(h\tau+k\tau, 2)\ {}_2F_1\left(h\tau+k\tau, k\tau+1; h\tau+k\tau+2; \left(\frac{1-z}{z}\right)^\gamma\right)$$

Thus, the integral in Equation (19), can be simplified as follows:

(1) $\Psi(1,1) = \tau^2\gamma z^{-(\tau\gamma+1)}(1-z)^{\tau\gamma-1}B(2\tau,2)\quad {}_2F_1\left(2\tau, \tau+1; 2\tau+2; 1-\left(\frac{1-z}{z}\right)^\gamma\right)$

(2) $\Psi(2,1) = \tau^2\gamma z^{-(\tau\gamma+1)}(1-z)^{\tau\gamma-1}B(3\tau,2)\quad {}_2F_1\left(3\tau, \tau+1; 3\tau+2; 1-\left(\frac{1-z}{z}\right)^\gamma\right)$

(3) $\Psi(1,2) = \tau^2\gamma z^{-(2\tau\gamma+1)}(1-z)^{2\tau\gamma-1}B(3\tau,2)\quad {}_2F_1\left(3\tau, 2\tau+1; 3\tau+2; 1-\left(\frac{1-z}{z}\right)^\gamma\right)$

(4) $\Psi(2,2) = \tau^2\gamma z^{-(2\tau\gamma+1)}(1-z)^{2\tau\gamma-1}B(4\tau,2)\quad {}_2F_1\left(4\tau, 2\tau+1; 4\tau+2; 1-\left(\frac{1-z}{z}\right)^\gamma\right)$

And by substituting the result of (1)-(4) to Equation (19), the results follow. □

Figure 4 shows the graph of Equation (17) for $\rho = 0.5$. Each plot contains three curves corresponding to selected values of γ and τ. The effect of the parameters is evident. Note that $\left(\frac{X}{X+Y}\right)$ is between 0 and 1. The graph shows the domain on [0, 1].

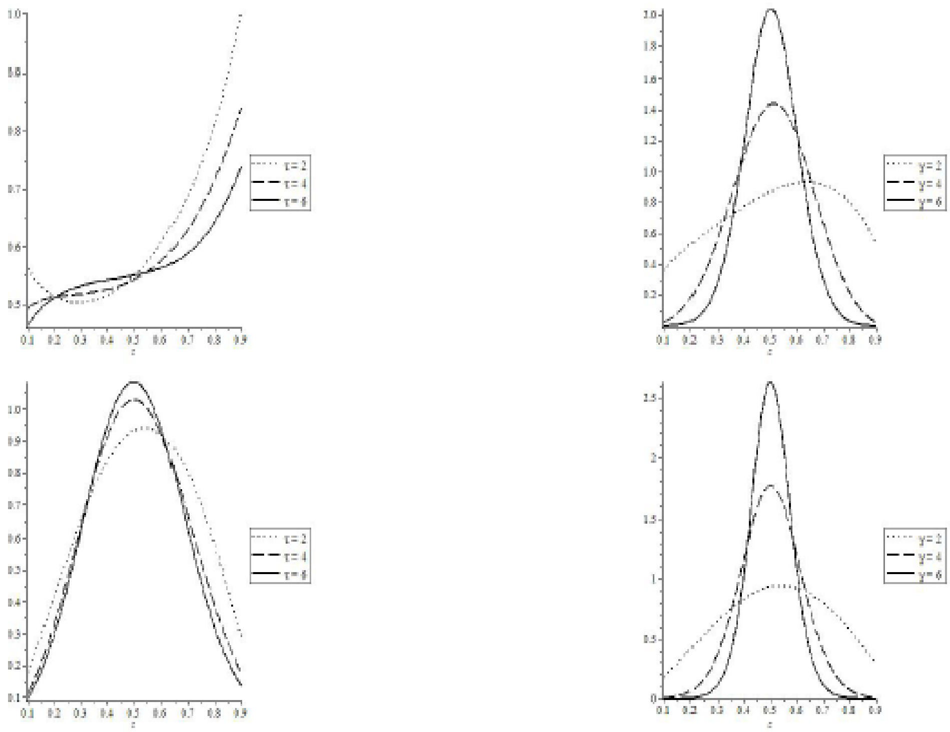

Figure 4: Graph of the pdf in (17)

3.0 Moments

Here we derive the of $V = XY$, $Z = \dfrac{X}{X+Y}$ and $W = \dfrac{X}{Y}$ when X and Y are distributed according to Equation (8).

Theorem 3.1. *If X and Y are jointly distributed according to Equation (8), then the (a,b)-th product moment of X and Y is given by*

$$\begin{aligned}\mu'_{a,b;\rho}(X,Y) = \theta^{a+b}\Gamma\Big(1-\frac{a}{\gamma}\Big)\Gamma\Big(1-\frac{b}{\gamma}\Big)\Bigg[\frac{\Gamma(\tau+\frac{a}{\gamma})\Gamma(\tau+\frac{b}{\gamma})}{\Gamma^2(\tau)} \\ +\rho\Bigg(\frac{\Gamma(\tau+\frac{a}{\gamma})}{\Gamma(\tau)}-\frac{\Gamma(2\tau+\frac{a}{\gamma})}{\Gamma(2\tau)}\Bigg)\Bigg(\frac{\Gamma(\tau+\frac{b}{\gamma})}{\Gamma(\tau)}-\frac{\Gamma(2\tau+\frac{b}{\gamma})}{\Gamma(2\tau)}\Bigg)\Bigg]\end{aligned} \tag{21}$$

where the real numbers a and b are such that $max\{a,b\} < \tau$ and $-1 < \rho < 1$.

Proof. By definition, the $\{a,b\}$-th moment of $f_{X,Y}(x,y;\theta,\tau,\gamma;\rho)$ is

$$\begin{aligned}\mu'_{a,b;\rho}(X,Y) &= \int_0^\infty\int_0^\infty x^a y^b f_{X,Y}(x,y;\theta,\tau,\gamma;\rho)dxdy \\ &= \int_0^\infty \frac{\tau\gamma(\frac{x}{\theta})^{\gamma\tau}}{x[1+(\frac{x}{\theta})^\gamma]^{\tau+1}}x^a dx\cdot\int_0^\infty \frac{\tau\gamma(\frac{y}{\theta})^{\gamma\tau}}{y[1+(\frac{y}{\theta})^\gamma]^{\tau+1}}y^b dy \\ &+\rho\Bigg[\int_0^\infty\Bigg(\frac{\tau\gamma(\frac{x}{\theta})^{\gamma\tau}}{x[1+(\frac{x}{\theta})^\gamma]^{\tau+1}}-2\frac{\tau\gamma(\frac{x}{\theta})^{\gamma\tau}}{x[1+(\frac{x}{\theta})^\gamma]^{\tau+1}}x^a\Bigg(\frac{(\frac{x}{\theta})^\gamma}{1+(\frac{x}{\theta})^\gamma}\Bigg)^\tau\Bigg)dx \\ &\int_0^\infty\Bigg(\frac{\tau\gamma(\frac{y}{\theta})^{\gamma\tau}}{y[1+(\frac{x}{\theta})^\gamma]^{\tau+1}}y^b-2\frac{\tau\gamma(\frac{y}{\theta})^{\gamma\tau}}{y[1+(\frac{y}{\theta})^\gamma]^{\tau+1}}y^b\Bigg(\frac{(\frac{y}{\theta})^\gamma}{1+(\frac{y}{\theta})^\gamma}\Bigg)^\tau\Bigg)dy\Bigg]\end{aligned}$$

For simplicity, we write

$$\mu'_{a,b;\rho}(X,Y) = \Psi(1)\cdot\Phi(1)+\rho\Big[(\Psi(1)-2\Psi(2))\cdot(\Phi(1)-2\Phi(2))\Big] \tag{22}$$

where $\Psi(h)$ and $\Phi(h)$, for $h\in\{1,2\}$ is defined as follows

$$\Psi(h) = \int_0^\infty \frac{\tau\gamma(\frac{x}{\theta})^{\gamma\tau}}{x[1+(\frac{x}{\theta})^\gamma]^{\tau+1}}x^a dx \quad\text{and}\quad \Phi(h) = \int_0^\infty \frac{\tau\gamma(\frac{y}{\theta})^{\gamma\tau}}{y[1+(\frac{y}{\theta})^\gamma]^{\tau+1}}y^b dy$$

Substituting $u = x^\gamma$, the integral $\Psi(h)$ can be written as

$$\begin{aligned}\Psi(h) &= \tau\theta^\gamma\cdot\lim_{b\to\infty}\int_0^b u^{h\tau+(a/\gamma)-1}(u+\theta^\gamma)^{-(h\tau+1)}du \\ &= \tau\theta\gamma\Big[(\theta^\gamma)^{(a/\gamma)-1}B\Big(h\tau+\frac{a}{\gamma},1-\frac{a}{\gamma}\Big)\Big] \\ &= \frac{\tau\theta^a\Gamma\Big(h\tau+\frac{a}{\gamma}\Big)\Gamma\Big(1-\frac{a}{\gamma}\Big)}{h\tau\Gamma(h\tau)}\end{aligned}$$

Similarly,

$$\Phi(h) = \int_0^\infty \frac{\tau\gamma(\frac{y}{\theta})^{h\tau\gamma}}{y[1+(\frac{y}{\theta})\gamma]^{h\tau+1}} y^b dy = \frac{\tau\theta^b\Gamma\Big(h\tau+\frac{b}{\gamma}\Big)\Gamma\Big(1-\frac{b}{\gamma}\Big)}{h\tau\Gamma(h\tau)}.$$

Then, the last equality follows directly from (22) and substituting the results which completes the proof. □

At this point we can now easily derive the raw moments of the random variables $V = XY$, $W = X/Y$ and $Z = X/(X+Y)$ when X and Y are jointly distributed according to (8) .

Theorem 3.2. *If X and Y are jointly distributed according to (8) then a-th raw moment of $V = XY$ is given by*

$$\mu'_{a;\rho}(V) = \theta^{2a}\Gamma^2\left(1-\frac{a}{\gamma}\right)\left[\frac{\Gamma^2\left(\tau+\frac{a}{\gamma}\right)}{\Gamma^2(\tau)} + \rho\left(\frac{\Gamma\left(\tau+\frac{a}{\gamma}\right)}{\Gamma(\tau)} - \frac{\Gamma\left(2\tau+\frac{a}{\gamma}\right)}{\Gamma(2\tau)}\right)^2\right] \tag{23}$$

Proof. Observed that

$$\mu'_{a;\rho}(V) = \mathbb{E}(V^a) = \mathbb{E}((XY)^a) = \mathbb{E}(X^aY^a).$$

By putting $b = a$ in Equations (21) then result follows directly. □

Theorem 3.3. *If X and Y are jointly distributed according to (8) then the a-th raw moment of $W = X/Y$ is given by*

$$\begin{aligned}\mu'_{a;\rho}(W) = \Gamma\left(1-\frac{a}{\gamma}\right)\Gamma\left(1+\frac{a}{\gamma}\right)&\left[\frac{\Gamma(\tau+\frac{a}{\gamma})\Gamma(\tau-\frac{a}{\gamma})}{\Gamma^2(\tau)}\right.\\ &\left.+\rho\left(\frac{\Gamma(\tau+\frac{a}{\gamma})}{\Gamma(\tau)} - \frac{\Gamma(2\tau+\frac{a}{\gamma})}{\Gamma(2\tau)}\right)\left(\frac{\Gamma(\tau-\frac{a}{\gamma})}{\Gamma(\tau)} - \frac{\Gamma(2\tau-\frac{a}{\gamma})}{\Gamma(2\tau)}\right)\right]\end{aligned} \tag{24}$$

Proof. Note that

$$\mu'_{a;\rho}(W) = \mathbb{E}(W^a) = \mathbb{E}((X/Y)^a) = \mathbb{E}\left(X^aY^{-a}\right).$$

Hence, by putting $b = -a$ in (21) then the result follows. □

Theorem 3.4. *If X and Y are jointly distributed according to (8) then the a-th raw moment of $Z = \dfrac{X}{X+Y}$ is given by*

$$\begin{aligned}\mu'_{a;\rho}(Z) = \sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k\Gamma\left(1-\frac{a}{\gamma}+k\right)\Gamma\left(1+\frac{a}{\gamma}+k\right)&\\ \left[\frac{\Gamma\left(\tau+\frac{a}{\gamma}+k\right)\Gamma\left(\tau-\frac{a}{\gamma)}+k\right)}{\Gamma^2(\tau)}\right.&\\ +\rho\left(\frac{\Gamma\left(\tau+\frac{a}{\gamma}+k\right)}{\Gamma(\tau)} - \frac{\Gamma\left(2\tau+\frac{a}{\gamma}+k\right)}{\Gamma(2\tau)}\right)&\\ \left.\left(\frac{\Gamma\left(\tau-\frac{a}{\gamma}+k\right)}{\Gamma(\tau)} - \frac{\Gamma\left(2\tau-\frac{a}{\gamma}+k\right)}{\Gamma(2\tau)}\right)\right]&\end{aligned} \tag{25}$$

Proof. Observe that ,

$$\begin{aligned}
\mu'_{a;\rho} = \mathbb{E}(Z^a) &= \mathbb{E}\left[\left(\frac{X}{X+Y}\right)^a\right] \\
&= \mathbb{E}\left[\left(\frac{X}{Y}\right)^a\left(1+\frac{X}{Y}\right)^{-a}\right] \\
&= \mathbb{E}\left[\left(\frac{X}{Y}\right)^a\sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k\left(\frac{X}{Y}\right)^k\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k\left(\frac{X}{Y}\right)^{a+k}\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k W^{a+k}\right] \\
&= \sum_{k=0}^{\infty}\binom{a-1+k}{k}(-1)^k\mathbb{E}\left[W^{a+k}\right]
\end{aligned}$$

The result follows directly by adding k to a in Equations (24). □

4.0 Conclusion

In this paper, we have derived the probability density functions of product and quotient of two random variables both having Inverse Burr Distribution. We also derived each corresponding *r th* raw moment. These moments are useful in the estimation of products or quotients of *X* and *Y*. Irregardless of the application setting of random variables, the results are expressed in terms of beta and hypergeometric functions. Hence, one can implement a code as these special functions and readily available in most common software.

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